# Macroscopic quantum phases of a deconfined QCD matter at finite density



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## Abstract

Formalism for a unified description of distinct superfluid phases of a deconfined QCD matter at finite density together with the phase of spontaneously broken chiral symmetry is presented. Dispersion laws of the quasiquark excitations in both diamagnetic and ferromagnetic phases with spontaneously broken chiral symmetry are exhibited explicitly.

At large enough baryon number densities the hadrons overlap. Due to the asymptotic freedom of QCD [1] in which we trust such a hadronic system should behave as a weakly interacting quark-gluon plasma [2]. Restricting ourselves to temperatures at which the thermal wavelength becomes comparable with interparticle spacing we can expect behavior of the system characteristic of a genuine quantum Fermi liquid.

In this paper we consider the chiral $SU(2)$ limit of QCD which is a good approximation to the real world within a 20% accuracy. Numerical value of the quark number density $n = \frac{2}{\pi^2}p_F^3$ at $T = 0$ at which the deconfinement sets in can be estimated by taking $n = 0.72 fm^{-3}$, the quark number density inside the nucleon. Coherence temperature at which the quantum behavior sets in is estimated as follows. Average energy $p$ of a massless quark equals $n_c n_f n_s 3\frac{1}{2}k_B T$ which defines the thermal wave length $\lambda_T = \frac{4}{9}\frac{\pi\hbar}{k_B T}$. It should be larger than $n^{-1/3}$, the average interquark spacing. Hence, below the line $T = \frac{4}{9}\pi n^{1/3}$ (with $\hbar = k_B$ set equal to one) in the $(n^{1/3}, T)$ QCD phase diagram we expect the behavior of the system characteristic of a genuine massless Landau Fermi liquid. For example, its specific heat $C_V \sim T$.

If at the Fermi surface of a Landau Fermi liquid there is an attraction between two fermions with momenta $\vec{p}_F$ and$-\vec{p}_F$ respectively, the system becomes unstable with respect to the spontaneous condensation of Cooper pairs. This happens in the electron Fermi liquids in metals, and turns that system into a BCS superconductor [3]. This happens also in the Fermi liquid of $^3$He, and turns that system into a chain of distinct superfluid phases [4]. It is clearly important to ask : what are the realistic interactions in the QCD Fermi liquid, whether it is of the Landau type, and whether the Cooper instability takes place also there.

Unfortunately, there are no experimental data at present, either real or the lattice ones which would check our considerations. There are, however, rather solid theoretical arguments

that to a reasonable approximation the effective interactions between massless quarks in the Landau QCD Fermi liquid are governed by an $SU(3)_c \times SU(2)_L \times SU(2)_R$ globally invariant local four-quark Lagrangian $\mathcal{L}_{int}$. The individual terms in $\mathcal{L}_{int}$ originate from : (i) massive (due to Debye screening) chromoelectric gluon exchange; (ii) massive (due to dynamical Higgs mechanism) chromomagnetic gluon exchange; (iii) quark interactions with instanton zero modes; (iv) dynamically generated massive collective excitation exchange. Since at least points (ii) and (iv) are not under a safe theoretical control it is necessary to have a method capable of analyzing the phase structure of the Landau QCD Fermi liquid for a general form of $\mathcal{L}_{int}$.

Origin of the Cooper-pair instability is understood at present within the effective field-theory framework both in nonrelativistic [5] and relativistic [6] interacting Fermi systems. The local four-fermion couplings listed above do contain the attractive channels necessary for Cooper pairing, and the renormalization group determines their running from the matching point to the infrared. This provides another good reason for developing a general method of analysis of the phase structure of a low-temperature system of many fermions carrying spin, flavor and color. With perturbative forces in mind the idea of color superfluidity was mentioned already in [2], and elaborated in [7]. Recent papers [8–10] deal already with local four-quark interactions discussed above.

With a general local four-fermion interaction between quarks carrying spin, isospin and flavor there can be four types of the superfluid ground-state condensates different from zero due to Pauli principle (here we ignore the alluring possibility of spontaneous parity violation within parity-conserving QCD) :

$$v_{(1)} = \langle \overline{\psi} A \tau_2 \gamma_5 \psi^{\mathcal{C}} \rangle \, , \tag{1}$$
$$v_{(2)} = \langle \overline{\psi} A \tau_3 \tau_2 \gamma_0 \gamma_3 \psi^{\mathcal{C}} \rangle \, , \tag{2}$$
$$v_{(3)} = \langle \overline{\psi} S \tau_3 \tau_2 \gamma_5 \psi^{\mathcal{C}} \rangle \, , \tag{3}$$
$$v_{(4)} = \langle \overline{\psi} S \tau_2 \gamma_0 \gamma_3 \psi^{\mathcal{C}} \rangle \tag{4}$$

Although we use the convenient Lorentz-covariant notation with $\psi^{\mathcal{C}} = C\overline{\psi}$ where $C$ is the charge-conjugation matrix it is clear that the only sacred property of the ground state here is the translation invariance. We contemplate four distinct macroscopic quantum phases :

1. Condensate $v_{(1)}$ with the color-antisymmetric Clebsch-Gordan (CG) matrix $A$ chosen as $A^{ab} = i\epsilon^{ab3} = -(\lambda_2)^{ab}$ corresponds to a ground-state expectation value of the order parameter $\Phi^c$. It describes a Lorentz scalar, isosinglet color triplet superfluid.

2. Condensate $v_{(2)}$ corresponds to a ground-state expectation value of the order parameter $\Phi^c_{I;\mu\nu}$. It describes a color-triplet superfluid which at the same time behaves as ordinary as well as flavor ferromagnet : $\Phi^c_{I;\mu\nu}$ is an isospin 1 Lorentz antisymmetric tensor field.

3. Condensate $v_{(3)}$ with the color-symmetric CG matrix $S$ chosen as $S^{ab} = \frac{1}{3}\delta^{ab} - \frac{1}{\sqrt{3}}(\lambda_8)^{ab}$ corresponds to a ground-state expectation value of the order parameter $\Phi^{ab}_I$. It describes a Lorentz scalar color-sextet superfluid which at the same time behaves as a flavor ferromagnet.

4. Condensate $v_{(4)}$ corresponds to a ground-state expectation value of the order parameter $\Phi^{ab}_{\mu\nu}$. It describes an isoscalar color-sextet superfluid which at the same time behaves as an ordinary ferromagnet.

Recent papers [8–10] are devoted predominantly to the first case with some estimates [8] made also of the fourth possibility.

It is important to realize that all terms (1-4) are invariant with respect to global chiral rotations. This implies that they cannot produce physical, chiral-symmetry-violating quark masses in the deconfined phase. There is, however, a ground-state quark-antiquark condensate

$$w = \langle\overline{\psi}\psi\rangle \tag{5}$$

which can do namely this, and it should be considered together with the quark-quark condensates (1-4). It is well established that at finite density the local four-quark interactions considered above can give rise to the quark masses [11,8] and, by virtue of the Goldstone theorem also to the massless pions.

In this Letter we present main steps of a generalization of the field-theory approach developed for superconductivity by Nambu [12], and for spontaneous breakdown of chiral symmetry in the nucleon-pion system by Nambu and Jona-Lasinio [13]. It gives almost immediately the dispersion laws of the true fermionic excitations as exemplified explicitly below. It enables the (numerical) analysis of the phase structure of the system for a given set of couplings, as will be clearly seen from the following. It also provides a systematic way of investigating the gapless collective Nambu-Goldstone excitations corresponding to all symmetries spontaneously broken by the condensates (1-5). This part of the program will be published separately.

A self-consistent perturbation theory is defined by its bilinear Lagrangian with all terms (1-5) taken into account :

$$\begin{aligned}\mathcal{L}'_0 \equiv \mathcal{L}_0 - \mathcal{L}_\Sigma - \mathcal{L}_\Delta = \overline{\psi}(i\not{\partial} + \mu\gamma_0)\psi - \overline{\psi}\Sigma\psi - \\ \frac{1}{2}[\overline{\psi}(A\gamma_5\tau_2\Delta_{(1)} + A\gamma_0\gamma_3\tau_3\tau_2\Delta_{(2)} + S\gamma_5\tau_3\tau_2\Delta_{(3)} + S\gamma_0\gamma_3\tau_2\Delta_{(4)})\psi^{\mathcal{C}} + H.c.]\end{aligned} \tag{6}$$

It is assumed that its ground state for $\Delta_{(i)}, \Sigma$ different from zero is energetically advantageous with respect to the naive ground state corresponding to $\mathcal{L}_0 = \overline{\psi}(i\not{\partial}+\mu\gamma_0)\psi$. Nonzero values of $\Delta_{(i)}$ and $\Sigma$ are eventually found by solving the gap equations expressing the condition that the lowest-order perturbative contribution of $\mathcal{L}'_{int} = \mathcal{L}_{int} + \mathcal{L}_\Delta + \mathcal{L}_\Sigma$, using the propagator defined by $\mathcal{L}'_0$, vanishes.

Main trick which enables to realize the program using the standard field-theory methods (first real insight into the problem was achieved by the BCS-like variational calculation [8]) consists of introducing the field

$$q^a_{\alpha A}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi^a_{\alpha A}(x) \\ P^{ab}T_{AB}\delta_{\alpha\beta}\psi^{\mathcal{C}}_{b\beta B}\end{pmatrix} \tag{7}$$

The matrices $P$ and $T$ defined as $P = e^{i\alpha}A + e^{i\sigma}S,\ T = (\cos\Theta + i\tau_3\sin\Theta)e^{i\phi}\tau_2$ have the property $P^+P = T^+T = 1$. The field $q$ operates in space of Pauli matrices abbreviated as $\Gamma_i$. In terms of $q$ the Lagrangian (6) of central interest is

$$\mathcal{L}_0' = \overline{q} \left[ \begin{array}{cc} \not p - \Sigma + \mu\gamma_0 & -\Delta \\ -\gamma_0\Delta^+\gamma_0 & \not p - \Sigma - \mu\gamma_0 \end{array} \right] q \equiv \overline{q} S^{-1}(q) q. \tag{8}$$

We have introduced

$$\Delta \equiv \Delta_S \gamma_5 + \Delta_V \gamma_0 \gamma_3$$

where

$$\Delta_S = \Delta_{(1)} e^{-i(\alpha+\phi)} \cos\Theta - i\Delta_{(3)} e^{-i(\sigma+\phi)} \sin\Theta$$

$$\Delta_V = \Delta_{(4)} e^{-i(\sigma+\phi)} \cos\Theta - i\Delta_{(2)} e^{-i(\alpha+\phi)} \sin\Theta$$

For rewriting any local four-quark interaction in terms of $q$ it is convenient first to symmetrize all bilinear combinations $\overline{\psi}...\psi$ with respect to $\psi$ and $\psi^{\mathcal{C}}$, and then to use $\psi = \frac{1}{\sqrt{2}}(1+\Gamma_3)q$, and $\psi^{\mathcal{C}} = \frac{1}{\sqrt{2}}(1-\Gamma_3)P^+T^+q$.

Calculation of the quasiquark propagator requires some manual work. Abbreviating

$$S(p) \equiv \begin{pmatrix} I & J \\ K & L \end{pmatrix}$$

and imposing the simplifying constraint $Re\Delta_S\Delta_V^* = 0$ we find

$$I = \frac{\not p_+ + \Sigma}{D_+}[1 + \Delta \frac{\not P + M}{D} \gamma_0 \Delta^+ \gamma_0 (\not p_+ + \Sigma)]$$

$$K = \frac{\not P + M}{D} \gamma_0 \Delta^+ \gamma_0 (\not p_+ + \Sigma)\,,$$

where

$$\begin{aligned}
p_+^\mu &\equiv ((p_0+\mu), \vec{p})\,,\ D_+ = (p_0+\mu)^2 - \epsilon_p^2\,, \\
\epsilon_p^2 &= \vec{p}^{\,2} + \Sigma^2\,,\ M = [D_+ - (|\Delta_S|^2 - |\Delta_V|^2)]\Sigma\,, \\
P_3 &= [D_+ - (|\Delta_S|^2 - |\Delta_V|^2)]p_3\,, \\
P_0 &= D_+(p_0 - \mu) - (|\Delta_S|^2 - |\Delta_V|^2)(p_0+\mu)\,, \\
P_1 &= [D_+ - (|\Delta_S|^2 + |\Delta_V|^2)]p_1 - 2|\Delta_S||\Delta_V|p_2\,, \\
P_2 &= [D_+ - (|\Delta_S|^2 + |\Delta_V|^2)]p_2 + 2|\Delta_S||\Delta_V|p_1\,, \\
D &= P^2 - M^2 = P_0^2 - (\vec{P}^{\,2} + M^2)\,.
\end{aligned} \tag{9}$$

Formulas for $J$ and $L$ follow from those for $K$ and $I$ by replacements $\pm\mu \leftrightarrow \mp\mu$, and $\Delta \leftrightarrow \gamma_0\Delta^+\gamma_0$.

For particular case $\Delta_V = 0$ considered in [8] (without $\Sigma$, i.e., with $\epsilon_p = |\vec{p}\,|$)

$$D_S = \left[(p_0+\mu)^2 - \epsilon_p^2\right]\left[p_0^2 - [(\mu+\epsilon_p)^2 + |\Delta_S|^2]\right]\left[p_0^2 - [(\mu-\epsilon_p)^2 + |\Delta_S|^2]\right] \tag{10}$$

We have verified (the $p_0$ integration in the gap equation is trivially done using Cauchy theorem) that for the instanton- mediated interaction the gap equation (4.4) of [8] is obtained in an approximation of keeping only the leading term of the strongest residue. We have also verified that in the vacuum sector ($\mu = 0$) where the formulas greatly simplify our coupled gap equations for $\Delta_{(1)}$ and $\Sigma$ are in accord with corresponding formulas of [14].

If the interaction is such that the solution $\Delta_S \neq 0, \Sigma \neq 0$ is energetically favorable, the lowest-order self-consistent perturbation theory (nonperturbative in the coupling constants) turns the system of interacting massless quarks into a system of noninteracting massive quasiquarks with the dispersion laws given by (10). The corresponding specific heat exhibits characteristic exponential behavior. In the next mandatory step the list of the physical excitations has to be supplementented with the gapless Nambu-Goldstone collective excitations interacting in a calculable manner with the massive quasiquarks.

Analogously, for the particular case $\Delta_S = 0$ we find the dispersion laws of the noninteracting quasiquarks in the form

$$\begin{aligned} D_V = {} & [(p_0+\mu)^2 - \epsilon_p^2] \\ & \left\{ p_0^2 - \epsilon_p^2 - \mu^2 + |\Delta_V|^2 - 2\left[(\mu^2 - |\Delta_V|^2)\epsilon_p^2 + (p_3^2 + \Sigma^2)|\Delta_V|^2\right]^{1/2} \right\} \\ & \left\{ p_0^2 - \epsilon_p^2 - \mu^2 + |\Delta_V|^2 + 2\left[(\mu^2 - |\Delta_V|^2)\epsilon_p^2 + (p_3^2 + \Sigma^2)|\Delta_V|^2\right]^{1/2} \right\} \end{aligned} \tag{11}$$

It nicely exhibits spontaneous breakdown of the rotational symmetry in this phase. The formula (11) enables straightforward quantitative analysis of the $\Delta_V$ formation (together with $\Sigma$), apparently quite involved in the variational approach. While the $dp_0$ integration remains trivial, the $d^3p$ one becomes involved.

Formalism presented above permits a general analysis of the phase structure of the deconfined QCD matter for any given interaction between quarks, i.e., not only the local four-fermion one. The constraint $Re\Delta_S\Delta_V^* = 0$ was used merely for an easy inversion of the general matrix

$$\not{P} - M + 2Re\Delta_S\Delta_V^*[p_3\gamma_0\gamma_5 - (p_0+\mu)\gamma_3\gamma_5 + \Sigma\sigma_{12}]$$

entering the quasiquark propagator $S$.

The model is defined together with a prescription of handling the three-momentum integrations. With the formfactor $F(p^2) = [\Lambda^2/(p^2+\Lambda^2)]^{1/2}$ suggested in [8] the dynamically generated quark mass falls down at large momenta within logarithmic accuracy in accordance with asymptotic freedom [15]. For obtaining realistic numerical estimates of $\Sigma, \Delta$ and $T_c$ the correct functional dependence of these quantities upon the dimensional parameters of the model may be crucial.

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